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Adjustment of Census Age Data

POPULATION by age and sex have important applications in demographic studies. In some planning problems they form the starting point. Census usually supplies such data for the country as a whole. Unfortunately, the observed data suffer from a number of errors, notable among them are omission and misstatement errors. Adjustment of age data is necessary before it can be used for any refined type of work. In this paper a possible approach for adjustments of age data due to the two types of errors will be discussed.

In developing countries age misstatements occur due to misunderstanding of the census question about age, illiteracy, ignorance, social and cultural attitudes of the respondents and a variety of other factors. Add to these the usual tendency to understate or overstate age for persons in certain age groups and we have a situation where no mathematical model can reasonably hope to explain the causes behind age misstatement. However, the effect of all these factors is limited in number. In every case of canvassing for age, one of the following events is likely to happen :

(i) the age is correctly stated;

(ii) the age is overstated;

(iii) the age is understated;

(iv) no age statement is given;

(v) the person is completely omitted by the respondent.

Item (iv), above, forms the usual "age not stated" group in age tables; this group will not be considered in the present work. If we assume that age data is available in the conventional five year age groups 0-4, 5-9, 10-14, . . . etc., the effect of (ii) and (iii) will amount to a shift of certain number of persons from its proper age group to neighbouring age groups.

The generally accepted method of smoothing age data whether in single years or in five year age groups, is to apply a suitable moving average to them. The resulting series is assumed to be free from the effects of age misstatement error; adjustments due to ommissions are then added to the averages and, finally, the figures are prorated back to the actual census count. This practically amounts to the assumption that omissions have no significant effect on the smoothed figures obtained as moving averages. Let us examine this method in more detail even at the cost of repeating some concepts from a textbook on time series analysis. The method of moving averages is a flexible and powerful tool, but for it to produce meaningful results, it is necessary that the actual observations satisfy certain conditions. The more important of these are : (i) The observed data should comprise of two parts—a systematic and an error component the two being additive or multiplicative in nature; (ii) the systematic component should be representable by a polynomial of an ordered variable (here age) over the extent of the moving average; and (iii) the errors should be independent of the systematic part, random in nature and, consequently, their sum over each extent is near zero. If these conditions are satisfied, each moving average will exactly measure the systematic part in the middle of the extent of the moving average. For age data item (i) may be assumed to be true and item (ii) will hold if the extent of the moving average is not too large but item (iii) does not hold at all. Let us consider a simple example : suppose we are applying a simple five point moving average to single year age data which has only misstatement errors and no omission errors. The first average will be for the age 2 and will be $(P_0 + P_1 + P_2 + P_3 + P_4)/5$, the next average will be $(P_1 + P_2 + P_3 + P_4 + P_5)/5$ giving smoothed population for age 3, etc. Here P_i = the census count at age "i". If we assume that the misstatement of age is never more than 1 year, the first average will have the effect of the error in age 4, the second average will have effect of errors in ages 1 and 5. Thus, we distort the estimate for age 2 by an error two years removed from that age when in the original data errors were only one year apart; the same is true for the next average we impose on age 3 the errors in ages 1 and 5. If in the original data shifts were more than 1 year, the averages will have still higher errors. For weighted moving average for the same extent we gain in the sense that the systematic part need not be a straight line; it can be a polynomial of degree 3 or less, but we may play havoc with the misstatement errors. Similar arguments apply if the original data is in 5 year age groups. There appears to be no way of estimating the systematic part directly. In this situation one may try to eliminate the systematic part and concentrate on the error part alone. Variate difference is suitable for that purpose. Again, assuming the systematic part to be a polynomial in age, each difference will reduce the degree of the polynomial by one and after a certain order of difference, we will be left with only the error part. However, this approach is bound to fail as, even if there are no errors due to age misstatements and omissions; it is usually not possible to represent age data by a polynomial in age over the whole range. Thus, we do not expect to find a certain order of difference after which all higher order differences are zero.

Taking into consideration these difficulties, it appears that a better approach will be to assume that the observed -data has a basic pattern. On the assumption that effect of migration is insignificant, departures from the basic pattern are due to variations in fertility and mortality from those inherent in the basic pattern and also due to the effects of the two errors mentioned above. In this paper we assume that the basic pattern is a suitable stable population. Then, using the method of linear programming, we find the minimum transfers necessary between ages such that a certain order of difference of the smoothed series satisfy a given pattern. The computational procedure allows one to assess the impact of omission errors on the amount of transfers necessary for smoothing. Thus, both errors can be treated simultaneously, and this is a distinct advantage over the existing methods. The final smoothed series is obtained by an heuristic approach.

2 The Model

It is assumed that the data is available in five year age group 0-4, 5-9, 10-14, .. ., and that the error due to age misstatement can result in a shift of certain number of persons from its proper age group to the immediately preceding and following age groups only. Let/$_{0)i}$/i,/2,. . ., denote the enumerated population in ages0-4,5-9,10-14, etc., and let A/o, M_{lt} Af$_2$, etc., denote the corresponding true populations. Let Cj denote the proportion shifting from one age group due to age misstatement alone, and A_t denote the number of persons omitted from enumeration. Table I shows in symbolic form the enumerated number and relationships between the actual and enumerated numbers in each age group.

In the last column of the table, $L_i f_i$ denotes the net loss/gain of an age group due to misstatement error. The particular pattern for L_i has been selected for *smoothing 1961 Census male population of India.* All L_i *must be positive;* inspection of the observed data gives the necessary guidance. However if an error is committed in selecting a particular L_i, the computational procedure itself will indicate the error.

Next, we construct a difference table for M_i. It should be kept in mind that the mode of differencing here is minus times the usual definition used in calculus of finite differences. Select a particular order of difference and assign a pattern to $\Delta^k M_0$, $\Delta^k M_1$, $\Delta^k M_2$, . . . , $\Delta^k M_j$, based on a suitable stable population. Only the pattern in the form $\Delta^k M_j \gtrless 0$ is necessary; this allows a lot of flexibility for the actual value of the right hand side in each equation. The order of difference (K) selected here is 3. One may select any other order. However, it should not be too large, as then only a few equations will be available for estimating *Li*. Table 2 shows *in symbolic form the pattern for* $\Delta^3 M_i$ *when the basic pattern* followed by M_i is a stable population [1] corresponding to level 11 with male $e_0^0 = 42.12$ and $r = .020$. Now we can state the problem in a form suitable for applying Linear programming technique. Namely, find L_i ($i = 0, 2, 3, \ldots 12$) such that the overall net transfer $X_0 = L_0 f_0 + L_2 f_2 + \ldots + L_{12} f_{12}$ is as small as possible and at the same time $\Delta^3 M_i$ satisfy the restrictions stipulated in Table 2.

3. Numerical Illustration

For the purpose of illustration we will try to smooth the 1961 male population of India as reported in social and cultural tables [2] by the Indian Census. Table 3 shows the problem in a format suitable for applying Dual Simplex (minimization) Alogrithim[3]. It may be noted that $\Delta^3 A_i$ are not included at this stage. Their effect on L_i can be evaluated after the optimum feasible solution is obtained. In dual simplex minimization alogrithim the starting basis as usual consist entirely of the added variables (P, H, K, . . . J) and the objective function X_0 has the value zero. The starting basic solution is infeasible as right hand side in almost every equation is negative. Infeasibility in the basis is removed one at a time by introducing new variables in the basis. The criteria for selecting new variable and the method of introducing it in the basis are :

(i) Select the row with maximum negative value on the right hand side.

(ii) Calculate ratios of the coefficients of the variable in row number zero and the selected row. Ignore all ratios with positive or zero denominators, select that variable for which the ratio is a minimum.

(iii) Once the variable to enter the basis has been selected apply Gauss-Jorden [4] method of elimination of that variable from all rows except the selected row.

(iv) Continue such iterations until all numbers in the right hand side are positive.

Table 3 shows the initial and also the final figures which give the optimum solution. If at any iteration, we find at least one negative right hand side but no negative coefficient for variable in the row corresponding to the largest negative right hand side, the problem is said to be unbounded; that is, there is no finite set of L_i which will satisfy all the restrictions on $\Delta^3 M_i$ and at the same time make X_0 a minimum. If this happens, it means that either we have selected a wrong basic stable population pattern or some of the L_i should be negative. Selection of appropriate basic pattern is relatively easy, so we need only change some of the L_i. For example, if suppose L_4 should be made negative to reach an optimum solution, we should replace

$$L_4 f_4 = C_7 M_4 - C_6 M_3 \text{ by } L_4' f_4 = C_6 M_3 - C_7 M_4.$$

4. Effect of Omission Errors on the Transfer Coefficients

Rows in the lower part of Table 3, which give the optimal solution when the non-basic variables are given zero values, were obtained by adding or substracting some multiple of a particular row in each iteration. Total number of iterations required to pass from the initial table (upper part of Table 3) to the final table, in this case, were seven. Now if $\Delta^3 A_i$ are not zero, their contribution to the final solution will be similar to those of the variables, $P, H, K, \ldots, J$. For example, if only $\Delta^3 A_0$ is a non-zero positive number and all other $\Delta^3 A_i$ are zero, then the final solution will give in :

$$\text{Row 0,} \quad X_0 = 98845.4 - \frac{1}{4} \Delta^3 A_0;$$

$$\text{Row 1,} \quad L_0 = .106161013 - \frac{1}{4 f_0} \Delta^3 A_0; \text{ and}$$

Rows 2, 3, 4, 5, 7, 8, 9 and 10 remain unchanged as coefficient of P in each happens to be zero.

$$\text{Row } 6\ H = 7474.8 + \frac{1}{4}\,\Delta^3 A_0$$

This means that if omission error is present only in age group 0-4, the net transfer necessary for 0-4 will be reduced by $\frac{1}{4}A_0$ and consequently the overall net transfer will be reduced by $\frac{1}{4}A_0$ (row 0 above). Using similar arguments it can be seen that if all $\Delta^3 A_i$ are positive, the reduced overall transfer and revised transfer coefficients will have values

$$X_0 = 98845.4 - \frac{1}{4}\,\Delta^3 A_0 + \frac{105}{159}\,\Delta^3 A_2 + \frac{102}{159}\,\Delta^3 A_3 - \frac{12}{53}\,\Delta^3 A_5 - \frac{45}{159}\,\Delta^3 A_7 - \frac{83}{265}\,\Delta^3 A_8 - \frac{87}{265}\,\Delta^3 A_9$$

$$L_0 = .106161013 - \frac{1}{4f_0}\,\Delta^3 A_0 + \frac{43}{159f_0}\,\Delta^3 A_2 + \frac{13}{159f_0}\,\Delta^3 A_3 - \frac{3}{212f_0}\,\Delta^3 A_5 + \frac{1}{318f_0}\,\Delta^3 A_7 + \frac{1}{795f_0}\,\Delta^3 A_8 + \frac{1}{3180f_0}\,\Delta^3 A_9$$

$$L_3 = .111322973 - 0 + \frac{43}{159f_3}\,\Delta^3 A_2 + \frac{13}{159f_3}\,\Delta^3 A_3 - \frac{3}{212f_3}\,\Delta^3 A_5 + \frac{1}{318f_3}\,\Delta^3 A_7 + \frac{1}{795f_3}\,\Delta^3 A_8 + \frac{1}{3180f_3}\,\Delta^3 A_9$$

$$L_6 = .108793916 - 0 + \frac{13}{159f_3}\,\Delta^3 A_2 + \frac{52}{156f_6}\,\Delta^3 A_3 - \frac{3}{53f_6}\,\Delta^3 A_5 + \frac{2}{159f_6}\,\Delta^3 A_7 + \frac{4}{795f_6}\,\Delta^3 A_8 + \frac{1}{795f_6}\,\Delta^3 A_9$$

$$L_{12} = .006945803 - 0 - \frac{4}{159f_{12}}\,\Delta^3 A_2 - \frac{16}{159f_{12}}\,\Delta^3 A_3 + \frac{5}{53f_{12}}\,\Delta^3 A_5 - \frac{74}{159f_{12}}\,\Delta^3 A_7 - \frac{125}{159f_{12}}\,\Delta^3 A_8 - \frac{71}{159f_{12}}\,\Delta^3 A_9$$

$$L_9 = .001351693 - 0 - \frac{2}{159f_9}\,\Delta^3 A_2 - \frac{8}{159f_9}\,\Delta^3 A_3 + \frac{5}{106f_9}\,\Delta^3 A_5 - \frac{37}{159f_9}\,\Delta^3 A_7 - \frac{74}{795f_9}\,\Delta^3 A_8 - \frac{37}{1590f_9}\,\Delta^3 A_9$$

$$L_7 = .093729352 - 0 + \frac{9}{159f_7}\,\Delta^3A_2 + \frac{36}{159f_7}\,\Delta^3A_3 - \frac{45}{212f_7}\,\Delta^3A_5$$
$$+ \frac{15}{318f_7}\,\Delta^3A_7 + \frac{3}{159f_7}\,\Delta^3A_8 + \frac{1}{212f_7}\,\Delta^3A_9,$$

$$L_{11} = .231612748 - 0 + \frac{3}{159f_{11}}\,\Delta^3A_2 + \frac{12}{159f_{11}}\,\Delta^3A_3 - \frac{15}{212f_{11}}\,\Delta^3A_5,$$
$$+ \frac{111}{318f_{11}}\,\Delta^3A_7 + \frac{429}{795f_{11}}\,\Delta^3A_8 + \frac{143}{1060f_{11}}\,\Delta^3A_9.$$

If some Δ^3A_i are negative, corresponding changes should be made in set (1). We come to an important conclusion; that if omission errors are present overall net transfer necessary to smooth the data is almost certainly going to be reduced when such errors are considered at the time of smoothing; also the net transfers necessary in each age are affected by omission errors in not only that age but of other ages also. The existing method of adding omissions to the smoothed figures and prorating back to the original census count ignores all these aspects. Smoothed population should be obtained by a procedure which modifies the observed census counts by as small an amount as possible; the present method works in that direction and, as such, should be recommended. We will not use these results in the present numerical illustration as 1961 Census of India does not supply the details of post-enumeration checks in a form suitable for calculating A_i.

5. Estimation of the Smoothed Population

Using L_i estimated in Table 3 or adjusted L_i from set 1, Section 4, one can calculate M_i from the relations given in Table 1. These M_i will satisfy the boundary conditions for Δ^3M_i and will not give M_i corresponding to the stable population taken as the basic pattern. Some of the basic variables L_0, L_3, L_6, L_{12}, L_9, L_7 and L_{11} can be changed without violating the restrictions on Δ^3M_i, but such changes will increase or decrease X_0. For example, L_3 can be reduced by δL_3 and still remain in the basis so long as δL_3 is less than $(636/53 \times 385)$; such a reduction in L_3 will reduce X_0 to $98845.4 - .111322973\ \delta L_3 f_3$. If reduction in L_3 is more than $(636/53 \times 385)$, it will go out of the optimal solution and its place will be taken by the variable L_5. This type of sensitivity analysis, though important for a comprehensive study, is not further pursued here. Instead, we will consider the possible values that the non-basic variables L_2, L_4, L_5, L_8 and L_{10} can take without violating the restrictions on Δ^3M_i. The reason for all these mathematical manipulations are apparent when we pick up the final smoothed series from all possible combinations of M_i. Now there are ten constraints on $\Delta^8 M_i$; as such we can get at most 10 non-zero L_i. The optimal basis contains seven non-zero L_i; so we can select at most 3 non-zero L_i from the non-basic variables L_2, L_4, L_5, L_8, and L_{10}. Out of these five variables, at the most, 3 can be selected in ${}^5C_1 + {}^5C_2 + {}^5C_3 = 25$ ways. Together with the minimum solution we can expect, at the most, 26 sets of M_i, each of which will satisfy the restrictions on $\Delta^3 M_i$. Fortunately, all the 25 combinations need not be calculated as some pairs of triplets of L_i cannot have simultaneously non-zero values. For example, in the present problem L_4 and L_5 cannot be simultaneously non-zero. Using rows 1, 2, 3, 4, 8, 9 and 10 of Table 3 (lower part), we have, after some calculations and reordering, the inequalities :

$$L_2 - .589008457\, L_4 - .347255815\, L_5 - .016941434\, L_8 + .001777605\, L_{10} \leqslant .109056087$$

$$-L_2 - 3.318221733\, L_4 - 1.578802606\, L_5 - .077024428\, L_8 + .008081905\, L_{10} \leqslant .291380134$$

$$-L_2 + 6.9263857\, L_4 + 13.61839482\, L_5 - 1.019092436\, L_8 + .106929827\, L_{10} \leqslant .809665327$$

$$-L_2 + 6.92638517\, L_4 + 11.46169684\, L_5 - 5.52008403\, L_8 + .579203235\, L_{10} \leqslant .857329898$$

$$-L_2 + 6.92638517\, L_4 + 11.46169684\, L_5 + 13.98421289\, L_8 - 23.97901395\, L_{10} \leqslant 2.468354244$$

$$L_2 - 6.92638517\, L_4 - 11.46169684\, L_5 - 12.76519433\, L_8 + 14.7696825\, L_{10} \leqslant .059970678$$

$$L_2 - 6.92638517\, L_4 - 11.46169684\, L_5 - 23.7363615\, L_8 - 12.85831182\, L_{10} \leqslant .039820571$$

Set (2) above, ensures that none of the basic variables L_0, L_3, L_6, L_7, L_9, L_{11}, and L_{12} is negative. Nine places of decimals have been kept in the coefficients as f_i are large eight figure numbers here; of course, we took both f_i and M_i in hundreds. From set (2) possible values for L_2, L_4, L_5, L_8, L_{10}, and their combinations are calculated, Substituting these values in appropriate rows of Table 3, the corresponding values for L_0, L_3, L_6, L_7, L_9, L_{11} and L_{12} are also calculated. Table 4 shows the L_i and $\Sigma L_i f_i = X_0$. Each set of L_i gave a set of M_i;

Table 5 shows the possible set of eighteen M_i. Now we come to the most crucial part of the analysis, namely, picking out the actual M_i from this jumble. This part of the work is less rigourous mathematically. Demographers conversant with the type of transition in Indian population can certainly give better guidelines than those given below. Anyway, let us proceed with the arguments. If the actual population really came from a stable population (level 11 e_0^0 42.12 and $r = .020$), then there is no ambiguity. The final smoothed population has to be a linear combination of the set in Table 5. In other words, $M_i = \sum_{j=1}^{18} \alpha_j M_{ij}$ with $\alpha_j \geqslant 0$, and $\Sigma\alpha_j = 1$ $(i = 0, 1, \ldots, 12)$. Finding α_j does not pose a serious problem, as we can use the concept that age score should be as small as possible, this will give us a set of relations for α_j from which actual α_j can be found. But we know that the actual Indian male population in 1961 was not a stable population of the above type. In the decades preceding 1961, fertility and mortality were both higher than those of the stable population. The results of slowly decreasing fertility and rather sharp decreases in mortality in recent years are an increase in the population in younger ages and a decrease in population in the higher ages, as compared to a stable population. This hypothesis suggests that we seek M_0, $M_1, \ldots, M_j$ among the larger few values in each row of Table 5, and M_{j+1}, $M_{j+2}, \ldots, M_{12}$ among the lower few values, with the added restriction that sum of M_i should equal the total population in 1961. This procedure is likely to give us the optimum solution when the actual 1961 population differs from stable pattern because, (i) each set of M_i in Table 5 satisfies the basic pattern, (ii) M_i are free from the effect of transfer errors due to age misstatement and omission errors, (if set 1 and corresponding set 2 were used to estimate L_i) and, (iii) the various combinations of L_i are in the neighbourhood of the minimum solution. In the present example j was taken as 6 corresponding to age 30-34 and smoothed M_6 was taken as an average of the eighteen possible M_6. Table 6 rearranges Table 5 such that M_0, $M_1, \ldots, M_5$ lie in order of magnitude, M_6 has the average value and M_7, $M_8, \ldots, M_{12}$ have the lowest value in each row. From Table 6, it can be seen that the sixth set has a total M_i more than the census count and the seventh set has a total less than the census count. So the actual smoothed population should lie somewhere between these two sets. Table 7 shows the smoothed population where in the second column M_0 to M_5 are averages of the M_0 to M_5 in the sixth and seventh columns of Table 6 with an additional adjustment in M_0 so that $\Sigma M_i = 2260511$. The third column shows the sixth column of Table 6 with adjustments in M_0, M_1 and M_2 only. Perhaps the third column of Table 7 gives a better estimate of the actual 1961 male population of India.

The computations necessary for this type of analysis are heavy. Fortunately, the entire work can be computerized. For solving the linear programming problem stated in the upper part of Table 3,1.B.M. 360 package programme was used and in less than a minute the lower part of the same table was available. Unfortunately, this package cannot be used for ranging analysis, indicated in set 2. Also, the package requires some modifications if effects of omission errors are to be estimated as in set 1.

References

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Appendix

TABLE 1

Age Group	*Enumerated Population*	*Actual Population*	*Prop. Shift.*	*Omissions*
0–4	f_0	M_0	$C_0\downarrow\ \uparrow C_1$	A_0
5–9	f_1	M_1	$C_2\downarrow\ \uparrow C_3$	A_1
10–14	f_2	M_2	$C_4\downarrow\ \uparrow C_5$	A_2
15–19	f_3	M_3	$C_6\downarrow\ \uparrow C_7$	A_3
20–24	f_4	M_4	$C_8\downarrow\ \uparrow C_9$	A_4
25–29	f_5	M_5	$C_{10}\downarrow\ \uparrow C_{11}$	A_5
30–34	f_6	M_6	$C_{12}\downarrow\ \uparrow C_{13}$	A_6
35–39	f_7	M_7	$C_{14}\downarrow\ \uparrow C_{15}$	A_7
40–44	f_8	M_8	$C_{16}\downarrow\ \uparrow C_{17}$	A_8
45–49	f_9	M_9	$C_{18}\downarrow\ \uparrow C_{19}$	A_9
50–54	f_{10}	M_{10}	$C_{20}\downarrow\ \uparrow C_{21}$	A_{10}
55–59	f_{11}	M_{11}	$C_{22}\downarrow\ \uparrow C_{23}$	A_{11}
60–64	f_{12}	M_{12}		A_{12}
65+	f_{13}	M_{13}		

TABLE 1 (*contd.*)

Age Group	*Relation Between Actual and Enumerated Population*
0–4	$M_0 = f_0 - C_1M_1 + C_0M_0 + A_0 = f_0 + L_0f_0 + A_0$
5–9	$M_1 = f_1 - C_0M_0 - C_3M_2 + (C_1 + C_2)\,M_1 + A_1 = f_1 - L_0f_0 - L_2f_2 + A_1$
10–14	$M_2 = f_2 - C_2M_1 - C_5M_3 + (C_3 + C_4)\,M_2 + A_2 = f_2 + L_2f_2 - L_3f_3 + A_2$
15–19	$M_3 = f_3 - C_4M_2 - C_7M_4 + (C_5 + C_6)\,M_3 + A_3 = f_3 + L_3f_3 - L_4f_4 + A_3$
20–24	$M_4 = f_4 - C_6M_3 - C_9M_5 + (C_7 + C_8)\,M_4 + A_4 = f_4 + L_4f_4 + L_5f_5 + A_4$
25–29	$M_5 = f_5 - C_8M_4 - C_{11}M_6 + (C_9 + C_{10})\,M_5 + A_5 = f_5 - L_5f_5 - L_6f_6 + A_5$
30–34	$M_6 = f_6 - C_{10}M_5 - C_{13}M_7 + (C_{11} + C_{12})\,M_6 + A_6 = f_6 + L_6f_6 - L_7f_7 + A_6$
35–39	$M_7 = f_7 - C_{12}M_6 - C_{15}M_8 + (C_{13} + C_{14})\,M_7 + A_7 = f_7 + L_7f_7 - L_8f_8 + A_7$
40–44	$M_8 = f_8 - C_{14}M_7 - C_{17}M_9 + (C_{15} + C_{16})\,M_8 + A_8 = f_8 + L_8f_8 - L_9f_9 + A_8$
45–49	$M_9 = f_9 - C_{16}M_8 - C_{19}M_{10} + (C_{17} + C_{18})\,M_9 + A_9 = f_9 + L_9f_9 - L_{10}f_{10} + A_9$
50–54	$M_{10} = f_{10} - C_{18}M_9 - C_{21}M_{11} + (C_{19} + C_{20})\,M_{10} + A_{10} = f_{10} + L_{10}f_{10} - L_{11}f_{11} + A_{10}$
55–59	$M_{11} = f_{11} - C_{20}M_{10} - C_{23}M_{12} + (C_{21} + C_{22})\,M_{11} + A_{11} = f_{11} + L_{11}f_{11} + L_{12}f_{12} + A_{11}$
60–64	$M_{12} = f_{12} - C_{22}M_{11} + C_{23}M_{12} + A_{12} = f_{12} - L_{12}f_{12} + A_{12}$

TABLE 2

$$\Delta^3 M_0 = \Delta^3 f_0 + 4L_0 f_0 + 6L_2 f_2 - 4L_3 f_3 + L_4 f_4 + \Delta^3 A_0 \geqslant 0$$

$$\Delta^3 M_1 = \Delta^3 f_1 - L_0 f_0 - 4L_2 f_2 + 6L_3 f_3 - 4L_4 f_4 - L_5 f_5 + \Delta^3 A_1 \geqslant 0$$

$$\Delta^3 M_2 = \Delta^1 f_2 + L_2 f_2 - 4L_3 f_3 + 6L_4 f_4 + 4L_5 f_5 + L_6 f_6 + \Delta^3 A_2 \leqslant 0$$

$$\Delta^3 M_3 = \Delta^3 f_3 + L_3 f_3 - 4L_4 f_4 - 6L_5 f_5 - 4L_6 f_6 + L_7 f_7 + \Delta^3 A_3 \leqslant 0$$

$$\Delta^3 M_4 = \Delta^3 f_4 + L_4 f_4 + 4L_5 f_5 + 6L_6 f_6 - 4L_7 f_7 + L_8 f_8 + \Delta^3 A_4 \geqslant 0$$

$$\Delta^3 M_5 = \Delta^3 f_5 - L_5 f_5 - 4L_6 f_6 + 6L_7 f_7 - 4L_8 f_8 + L_0 f_9 + \Delta^3 A_5 \geqslant 0$$

$$\Delta^3 M_6 = \Delta^3 f_6 + L_6 f_6 - 4L_7 f_7 + 6L_8 f_8 - 4L_9 f_9 + L_{10} f_{10} + \Delta^3 A_6 \leqslant 0$$

$$\Delta^3 M_7 = \Delta^3 f_7 + L_7 f_7 - 4L_8 f_8 + 6L_9 f_9 - 4L_{10} f_{10} + L_{11} f_{11} + \Delta^3 A_7 \geqslant 0$$

$$\Delta^3 M_8 = \Delta^3 f_8 + L_8 f_3 - 4L_9 f_9 + 6L_{10} f_{10} - 4L_{11} f_{11} - L_{12} f_{12} + \Delta^3 A_8 \geqslant 0$$

$$\Delta^3 M_9 = \Delta^3 f_9 + L_9 f_9 - 4L_{10} f_{10} + 6L_{11} f_{11} + 4L_{12} f_{12} + \Delta^3 A_9 \geqslant 0$$

TABLE 3—VARIABLES AND THEIR COEFFICIENTS (Upper Part)

Row No.	*Basis and* X_0	L_0	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}
0	X_0	$-f_0$	$-f_2$	$-f_3$	$-f_4$	$-f_5$	$-f_6$	$-f_7$	$-f_8$	$-f_9$	$-f_{10}$	$-f_{11}$	$-f_{12}$
1	P	$-4f_0$	$-6f_2$	$4f_3$	$-f_4$	0	0	0	0	0	0	0	0
2	H	f_0	$4f_2$	$-6f_3$	$4f_4$	f_5	0	0	0	0	0	0	0
3	K	0	f_2	$-4f_3$	$6f_4$	$4f_5$	f_6	0	0	0	0	0	0
4	S	0	0	f_3	$-4f_4$	$-6f_5$	$-4f_6$	f_7	0	0	0	0	0
5	T	0	0	0	$-f_4$	$-4f_5$	$-6f_6$	$4f_7$	$-f_8$	0	0	0	0
6	Q	0	0	0	0	f_5	$4f_6$	$-6f_7$	$4f_8$	$-f_9$	0	0	0
7	R	0	0	0	0	0	f_6	$-4f_7$	$6f_8$	$-4f_9$	f_{10}	0	0
8	M	0	0	0	0	0	0	$-f_7$	$4f_8$	$-6f_9$	$4f_{10}$	$-f_{11}$	0
9	N	0	0	0	0	0	0	0	$-f_8$	$4f_9$	$-6f_{10}$	$4f_{11}$	$-f_{12}$
10	J	0	0	0	0	0	0	0	0	$-f_9$	$4f_{10}$	$-6f_{11}$	$-4f_{12}$

Table 3—Upper Part (*contd.*)

Row No.	*Basis and X_0*	*P*	*H*	*K*	*S*	*T*	*Q*	*R*	*M*	*N*	*J*	*Right-hand Side*
0	X_0	0	0	0	0	0	0	0	0	0	0	0
1	*P*	1	0	0	0	0	0	0	0	0	0	−58109
2	*H*	0	1	0	0	0	0	0	0	0	0	−81521
3	*K*	0	0	1	0	0	0	0	0	0	0	−65424
4	*S*	0	0	0	1	0	0	0	0	0	0	−36119
5	*T*	0	0	0	0	1	0	0	0	0	0	−30392
6	*Q*	0	0	0	0	0	1	0	0	0	0	− 7061
7	*R*	0	0	0	0	0	0	1	0	0	0	−16999
8	*M*	0	0	0	0	0	0	0	1	0	0	−25777
9	*N*	0	0	0	0	0	0	0	0	1	0	49870
10	*J*	0	0	0	0	0	0	0	0	0	1	−75138

TABLE 3 (Lower Part)

Row No.	Basis and X_0	L_0	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}
0	X_0	0	$-\frac{16}{106}f_2$	0	$-\frac{455f_4}{212}$	$-\frac{f_5}{53}$	0	0	$-\frac{722f_8}{265}$	0	$-\frac{249f_{10}}{159}$	0	0
1	L_0	1	$\frac{391f_2}{318f_0}$	0	$-\frac{665f_4}{636f_0}$	$-\frac{385f_5}{636f_0}$	0	0	$-\frac{12f_8}{265f_0}$	0	$-\frac{f_{10}}{159f_0}$	0	0
2	L_3	0	$-\frac{43f_2}{159f_3}$	1	$-\frac{206f_4}{159f_3}$	$-\frac{385f_5}{636f_3}$	0	0	$-\frac{12f_8}{265f_3}$	0	$\frac{f_{10}}{159f_3}$	0	0
3	L_6	0	$-\frac{13f_2}{159f_6}$	0	$\frac{130f_4}{159f_6}$	$\frac{251f_5}{159f_6}$	1	0	$-\frac{48f_8}{265f_6}$	0	$\frac{4f_{10}}{159f_6}$	0	0
4	L_{12}	0	$\frac{4f_2}{159f_{12}}$	0	$-\frac{40f_4}{159f_{12}}$	$-\frac{65f_5}{159f_{12}}$	0	0	$-\frac{37f_8}{53f_{12}}$	0	$\frac{170f_{10}}{159f_{12}}$	0	1
5	T	0	$-\frac{42f_2}{159}$	0	$\frac{261f_4}{159}$	$\frac{285f_5}{159}$	0	0	$-\frac{167f_8}{265}$	0	$-\frac{36f_{10}}{159}$	0	0
6	H	0	$\frac{365f_2}{318}$	0	$-\frac{1735f_4}{636}$	$-\frac{1289f_5}{636}$	0	0	$-\frac{12f_8}{53}$	0	$\frac{5f_{10}}{159}$	0	0
7	R	0	$-\frac{15f_2}{159}$	0	$-\frac{150f_4}{159}$	$\frac{204f_5}{159}$	0	0	$\frac{46f_8}{53}$	0	$-\frac{240f_{10}}{159}$	0	0
8	L_9	0	$\frac{2f_2}{159f_9}$	0	$-\frac{20f_4}{159f_9}$	$-\frac{65f_5}{318f_9}$	0	0	$-\frac{172f_8}{265f_9}$	1	$-\frac{74f_{10}}{159f_9}$	0	0
9	L_7	0	$-\frac{3f_2}{53f_9}$	0	$\frac{90f_4}{159f_7}$	$\frac{195f_5}{212f_7}$	0	1	$-\frac{36f_8}{53f_7}$		$\frac{15f_{10}}{159f_7}$	0	0
10	L_{11}	0	$-\frac{1f_2}{53f_{11}}$	0	$\frac{10f_4}{53f_{11}}$	$\frac{65f_5}{213f_{11}}$	0	0	$\frac{152f_8}{265f_{11}}$	0	$-\frac{207f_{10}}{159f_{11}}$	1	0

Table 3—Lower Part (*contd.*)

Row No.	Basis and X_0	P	H	K	S	T	Q	R	M	N	J	Right hand Side
0	X_0	$-\frac{1}{4}$	0	$-\frac{105}{159}$	$-\frac{102}{159}$	0	$-\frac{12}{53}$	0	$-\frac{45}{159}$	$-\frac{83}{265}$	$-\frac{87}{265}$	98845.4
1	L_0	$-\frac{1}{4f_0}$	0	$-\frac{43}{159f_0}$	$-\frac{13}{159f_0}$	0	$-\frac{3}{212f_0}$	0	$\frac{1}{318f_0}$	$\frac{1}{795f_0}$	$\frac{1}{3180f_0}$	.106161013
2	L_3	0	0	$-\frac{43}{159f_3}$	$-\frac{43}{159f_3}$	0	$-\frac{3}{212f_3}$	0	$\frac{1}{318f_3}$	$\frac{1}{795f_3}$	$\frac{1}{3180f_3}$	.111322973
3	L_6	0	0	$-\frac{13}{159f_6}$	$-\frac{52}{159f_6}$	0	$\frac{3}{53f_6}$	0	$\frac{2}{159f_6}$	$\frac{4}{795f_6}$	$\frac{1}{795f_6}$	.108793916
4	L_{12}	0	0	$\frac{4}{159f_{12}}$	$\frac{16}{159f_{12}}$	0	$\frac{5}{53f_{12}}$	0	$-\frac{74}{159f_{12}}$	$-\frac{125}{159f_{12}}$	$-\frac{71}{159f_{12}}$	.006945803
5	T	0	0	$-\frac{42}{159}$	$-\frac{168}{159}$	1	$\frac{27}{53}$	0	$-\frac{18}{59}$	$-\frac{36}{795}$	$-\frac{3}{265}$	22966.9
6	H	$\frac{1}{4}$	1	$-\frac{215}{159}$	$-\frac{65}{159}$	0	$-\frac{15}{212}$	0	$\frac{5}{318}$	$\frac{1}{159}$	$\frac{1}{636}$	7474.8
7	R	0	0	$-\frac{15}{159}$	$-\frac{60}{159}$	0	$-\frac{32}{52}$	1	$-\frac{120}{159}$	$-\frac{48}{159}$	$-\frac{12}{159}$	17136.2
8	L_9	0	0	$\frac{2}{159f_9}$	$\frac{8}{159f_9}$	0	$\frac{5}{106f_9}$	0	$-\frac{37}{159f_9}$	$-\frac{74}{795f_9}$	$-\frac{37}{1590f_9}$	.001351693
9	L_7	0	0	$-\frac{9}{159f_7}$	$-\frac{36}{159f_7}$	0	$-\frac{45}{212f_7}$	0	$\frac{15}{318f_7}$	$-\frac{3}{159f_7}$	$-\frac{1}{212f_7}$	.093729352
10	L_{11}	0	0	$-\frac{3}{159f_{11}}$	$-\frac{12}{149f_{11}}$	0	$-\frac{15}{212f_{11}}$	0	$\frac{111}{318f_{11}}$	$-\frac{429}{795f_{11}}$	$-\frac{143}{1060f_{11}}$	.231612748

TABLE 4—OTHER FEASIBLE SOLUTIONS

	Min. Sol.	*I*	*II*	*III*	*IV*	*V*
L_1	.106161013	0	0	.067397546	.173185819	.126258616
L_2	0	.111475919	.114620703	.039820571	0	0
L_3	.111322973	.159102198	.167085762	.126536585	.259516474	.147184791
L_4	0	.004050789	.009444962	0	.116895798	0
L_5	0	0	0	0	0	.0594538
L_6	.10879316	.120254287	.115394579	.11414381	0	0
L_7	.093729352	.103957664	.099062213	.098082817	.005211028	.019229369
L_8	0	.001836642	0	0	0	0
L_9	.001351693	0	0	0	.02883545	.024482843
L_{10}	0	0	.000729336	0	0	0
L_{11}	.231612748	.237030151	.237870476	.235349226	.155639533	.167671147
L_{12}	.00694583	0	0	.002333813	.100721227	.085870338
$\Sigma L_i f_i$	98845	105731	107469	100523	144505	99053

Table 4 (*contd.*)

	VI	*VII*	*VIII*	*IX*	*X*	*XI*
L_0	.109071963	.106153987	0	0	0	0
L_2	0	0	.115493736	.111222021	.109892036	.110927679
L_3	.116517222	.111310436	.169297822	.157573257	.154754926	.156949527
L_4	0	0	.010924948	0	0	0
L_5	0	0	0	.006229329	.002309737	.00538544
L_6	.132964219	.108735577	.114144938	.11233972	.119584933	.113833915
L_7	.200252205	.093472238	.09808312	.09808312	.103957657	.099062213
L_8	.176510059	0	0	0	.001836642	0
L_9	.143569566	.003123931	0	0	0	0
L_{10}	0	.00406039	0	0	0	.000729336
L_{11}	0	.240748704	.235349485	.235349486	.237029028	.237870476
L_{12}	.267909675	0	.002333465	.002333464	0	0
$\Sigma L_i f_i$	156971	99426	107980	103564	104090	103643

Table 4 (*contd.* on page 286)

Table 4 (*contd.*)

	XII	*XIII*	*XIV*	*XV*	*XVI*	*XVII*
L_0	.024989944	.05826721	.184335618	.172567227	.131287965	.12603851
L_2	.083415769	.0491986	0	0	0	0
L_3	.143246355	.130117248	.283391086	.258191907	.156158387	.146790332
L_4	0	0	.133161954	.115993349	0	0
L_5	0	0	0	0	.068461235	.059060176
L_6	.120253151	.115393447	0	0	0	0
L_7	.103957355	.099061906	.0596129	.002192804	.080601087	.016564127
L_8	.001836642	0	.110554899	0	.12039604	0
L_9	0	0	.121736251	.054137653	.124991025	.046105521
L_{10}	0	.000729336	0	.058456188	0	.049894906
L_{11}	.237029886	.237870341	0	.287753362	0	.280356498
L_{12}	0	0	.277221665	0	.275832333	0
$\Sigma L_i f_i$	102965	101022	187263	152509	138731	106185

TABLE 5

Age Group		*Min. Sol.*	*I*	*II*	*III*	*IV*	*V*
0–4	M_0	367104	331872	331872	354239	389348	373774
5–9	M_1	295512	301454	300628	297914	273268	288842
10–14	M_2	242041	262445	261786	249675	214479	235372
15–19	M_3	206692	214841	215344	209521	212980	213361
20–24	M_4	181988	182725	183707	181988	203262	193006
25–29	M_5	167931	166098	166875	167075	185324	174306
30–34	M_6	164518	164960	164849	164782	159167	157260
35–39	M_7	148787	149956	149512	149379	136745	138652
40–44	M_8	120733	121087	120865	120865	118057	118481
45–49	M_9	97496	97364	97298	97364	100172	99744
50–54	M_{10}	79073	78787	78809	78876	83087	82452
55–59	M_{11}	65466	65356	65400	65400	66804	66592
60+	$M_{12}+$	*123170*	*123566*	*123566*	*123433*	*117818*	*118665*
Total		2260511	2260511	2260511	2260511	2260511	2260511

Table 5 (*contd.* to p. 288)

Table 5 (*contd.*)

Age Group		*VI*	*VII*	*VIII*	*IX*	*X*	*XI*
0–4	M_0	368070	367102	331872	331872	331872	331872
5–9	M_1	294546	295514	300398	301521	301870	301598
10–14	M_2	241075	242044	261605	262662	262838	262701
15–19	M_3	207658	206689	215486	215294	214769	215178
20–24	M_4	181988	181988	183976	183142	182416	182986
25–29	M_5	164066	167940	167075	166210	165777	166127
30–34	M_6	153892	164544	164782	164493	164853	164599
35–39	M_7	141944	148752	149379	149379	149956	149512
40–44	M_8	128221	120561	120865	120865	121087	120865
45–49	M_9	111342	97297	97364	97364	97364	97298
50–54	M_{10}	91310	78962	78876	78876	78787	78809
55–59	M_{11}	68123	65552	65400	65400	65356	65400
60+	$M_{12}+$	108276	123566	123433	123433	123566	123566
Total		2260511	2260511	2260511	2260511	2260511	2260511

Table 5 (*contd.*)

Age Group		*XII*	*XIII*	*XIV*	*XV*	*XVI*	*XVII*
0—4	M_0	340165	351210	393048	389142	375443	373701
5—9	M_1	300534	298480	269568	273474	287173	288915
10—14	M_2	258021	251473	210039	214726	233703	235445
15—19	M_3	212629	210187	214460	212898	215030	213288
20—24	M_4	181988	181987	205222	203097	194676	192933
25—29	M_5	166098	166875	185324	185324	172636	174379
30—34	M_6	164960	164849	151766	159578	148911	157623
35—39	M_7	150178	149512	130784	136334	132449	138289
40—44	M_8	120865	120865	122374	115594	123247	116376
45—49	M_9	97360	97298	109217	97297	109534	97297
50—54	M_{10}	78786	78809	91310	81445	91310	81054
55—59	M_{11}	65356	65400	68654	68036	68575	67645
60+	M_{12}^{+}	123566	123566	107745	123566	107824	123566
Total		2260511	2260511	2260511	2260511	2260511	2260511

TABLE 6

Age Group		*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*
0–4	M_0	393048	389348	389142	375443	373774	373701
5–9	M_1	301870	301598	301521	301454	300628	300534
10–14	M_2	262838	262701	262622	262445	261786	261605
15–19	M_3	215486	215344	215294	215178	215030	214841
20–24	M_4	206222	203262	203097	194676	193006	192933
25–29	M_5	185324	185324	185324	174379	174306	172636
30–34	M_6	161133	161133	161133	161133	161133	161133
35–39	M_7	130784	130784	130784	130784	130784	130784
40–44	M_8	115594	115594	115594	115594	115594	115594
45–49	M_9	97298	97298	97298	97298	97298	97298
50–54	M_{10}	78787	78787	78787	78787	78787	78787
55–59	M_{11}	65358	65358	65358	65358	65358	65358
60+	M_{12}^{+}	107824	107824	107824	107824	107824	107824
Total		2321566	2314355	2313818	2280353	2275308	2273028

Table 6 (*contd.*)

Age Group		(*7*)	(*8*)	(*9*)	(*10*)	(*11*)	(*12*)
0–4	M_0	368070	367104	367102	354239	351210	340165
5–9	M_1	300398	298480	297914	295514	295512	294546
10–14	M_2	258021	251473	249675	242044	242041	235445
15–19	M_3	214769	214460	213361	213288	212890	212898
20–24	M_4	183976	183707	183142	182968	182725	182416
25–29	M_5	167940	167931	167075	167075	166875	166875
30–34	M_6	161133	161133	161133	161133	161133	161133
35–39	M_7	130784	130784	130784	130784	130784	130584
40–44	M_8	115594	115594	115594	115594	115594	115594
45–49	M_9	97298	97298	97298	97298	97298	97298
50–54	M_{10}	78787	78787	78787	78787	78787	78787
55–59	M_{11}	65358	65358	65358	65358	65358	65358
60+	M_{12}^{+}	107824	107824	107824	107824	107824	107824
Total		2249952	2239933	2235047	2211906	2208121	2189123

Table 6 (*contd.* on page 292)

Table 6 (*contd.*)

Age Group		*(13)*	*(14)*	*(15)*	*(16)*	*(17)*	*(18)*
0–4	M_0	331872	331872	331872	331872	331872	331872
5–9	M_1	288915	288842	287173	273474	273268	269568
10–14	M_2	235445	235372	233703	214726	214479	210039
15–19	M_3	212629	210187	209521	207658	206692	206689
20–24	M_4	181988	181988	181988	181988	181988	181988
25–29	M_5	166210	166127	166098	166098	165777	164066
30–34	M_6	161133	161133	161133	161133	161133	161133
35–39	M_7	130784	130784	130784	130784	130784	130784
40–44	M_8	115594	115594	115594	115594	115594	115594
45–49	M_9	97298	97298	97298	97298	97298	97298
50–54	M_{10}	78787	78787	78787	78787	78787	78787
55–59	M_{11}	65358	65358	65358	65358	65358	65358
60+	$M_{12}+$	107824	107824	107824	107826	107824	107824
Total		2173837	2171166	2167133	2132596	2130854	2121000

TABLE 7

Age Group	*Obs. Population in 000*	*Smoothed Population in 00* (2)	(3)
0–4	331872	370345	366763
5–9	330744	300027	297551
10–14	262746	259813	259009
15–19	185987	214805	214841
20–24	181988	188455	192933
25–29	185324	170288	172636
30–34	159876	161133	161133
35–39	136036	130784	130784
40–44	120865	115594	115594
45–49	97364	97298	97298
50–54	91310	78787	78787
55–59	52833	65358	65358
60+	123566	107824	107824
Total	2260511	2260511	2260511